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Shape optimization in contact problems of the theory of elasticity with incomplete external loading data☆

N.V. Banichuk*, S.Yu. Ivanova

Moscow, Russia

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ABSTRACT

The contact interaction without friction of an absolutely rigid punch with an elastic half-space is considered. The external loads on the elastic medium are not fixed in advance, but a set containing all the admissible forms of applied forces is assumed to be specified. Using a guaranteed (minimax) approach, problems of optimizing the shape of the punch from the condition that its mass is a minimum are formulated. Inequality-type constraints, imposed on the total force and moments applied to the punch from the elastic-medium side, are assumed. Using Betti's reciprocal theorem and calculating the "worst" case for different types of constraints, the corresponding forces are determined and the optimum shape of the punch is obtained in analytical form.

In problems of optimizing structures, the loading scenario is usually assumed to be known. In particular, the regions of application of the loads, the form of the distribution of the forces and their value are assumed to be specified. No changes are permitted in the specification of the external loads, and the optimum solutions determined under these conditions are sensitive to a change in the problem parameters. However, the situation is quite different in many applications: the regions where the loads act, the distribution of the applied forces and their limiting values are indeterminate and depend on a number of random quantities. To describe the uncertainties that arise and to formulate optimization problems with incomplete information, different approaches can be employed, for example, a probability approach, based on a specification of the probability density function of the random loads. The limited nature of this approach is the fact that, in many applications, this function is unknown. There is another approach, which does not use the probability density function; this is a guaranteed approach, based on a minimax description and on a determination of the unknown parameters in a calculation of the "worst" case. In this approach, the applied "indeterminate" loads do not need to be described using probability characteristics, they are not described by a probability density distribution, and pertain to a set of admissible loads. This approach enables a deterministic specification of the uncertainties to be used and enables modelling and optimization methods to be developed.[1,2]

Below we present methods of optimizing the shape of a rigid punch, indented into an elastic medium under quasistatic conditions. It is assumed that the forces applied to the elastic medium are not fixed, but belong to a specified set of external forces. A minimax formulation of the optimization problem is used, which assumes the "worst" scenario for determining the forms of the external loads, which belong to a specified set of admissible forces, and a search is made for the best solution which minimizes the quality functional with the limitations formulated.

1. Fundamental relations and formulation of the optimization problem

We will consider, in a rectangular system of coordinates Oxyz, the equilibrium of a punch in the form of an absolutely rigid shell, in contact without friction with an elastic half-space. The boundary xOy of the elastic half-space $z \geq 0$ is a combination of the contact region Ω_f (the base of the punch), the region where the external loads Ω_q are applied, and the region Ω_0, which is load-free. The punch surface (the punch shape) is given by the equation

$$z = f(x,y), \quad (x,y) \in \Omega_f \tag{1.1}$$

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* Corresponding author.
E-mail address: banichuk@ipmnet.ru (N.V. Banichuk).

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while the area of the surface is given approximately by the expression

$$S = \int_{\Omega_f} \sqrt{1+(\nabla f)^2}\,d\Omega_f = S_f + \frac{1}{2}\int_{\Omega_f} (\nabla f)^2\,d\Omega_f \tag{1.2}$$

where S_f is the area of the region Ω_f. The expression on the right-hand side of Eq. (1.2) is written on the assumption that the depth of indentation of the punch is small. For a known pressure distribution $p(x, y)((x, y) \in \Omega_f)$ the resulting force P acting on the punch and the moments M_x and M_y about the x and y axes are given by the expressions

$$P = \int_{\Omega_f} p(x,y)\,d\Omega_f, \quad M_x = \int_{\Omega_f} yp(x,y)\,d\Omega_f, \quad M_y = \int_{\Omega_f} xp(x,y)\,d\Omega_f \tag{1.3}$$

where p(x,y) is the pressure under the punch base.

The external forces applied to the region Ω_q, reduce to the following boundary conditions:

$$\sigma_{xz} = q_x(x,y), \quad \sigma_{yz} = q_y(x,y), \quad \sigma_{zz} = q_z(x,y), \quad (x,y) \in \Omega_q \tag{1.4}$$

where σ_{xz}, σ_{yz} and σ_{zz} are the of the stress tensor components, while $q_x(x,y)$, $q_y(x,y)$ and $q_z(x,y)$ are functions, defined in the region Ω_q. The specific form of these functions is unknown in advance, while the set Λ_q, which contains all the acceptable forms of the external forces, is assumed to be given, i.e.,

$$\{q_x,\ q_y,\ q_z\} \in \Lambda_q \tag{1.5}$$

We can consider as the Λ_q the continuous set, given by the inequalities

$$q_i(x,y) \ge 0, \quad (x,y) \in \Omega_q; \quad \int_{\Omega_q} q_i(x,y)\,d\Omega_q \le Q_i^*, \quad i = x,y,z \tag{1.6}$$

where $Q_i^* \ge 0$ are specified positive constants. According to inequalities (1.6) any unidirectional forces, the resultants of which do not exceed specified values, can be applied to the surface Ω_q. Another method of describing the set Λ_q consists of a discrete specification of the set of possible forms of the forces, i.e.,

$$\Lambda_q = \left\{q_i : q_i = q_i^j(x,y), \quad (x,y) \in \Omega_q, \quad i = x,y,z; \quad j = 1,2,\ldots,n\right\} \tag{1.7}$$

where $q_i^j(x, y)$ are specified functions, defining different cases of the loading of the surface Ω_q.

In the optimization problem considered here, the shape of the punch, i.e., the function $z = f(x,y)$, plays the role of the projection variable (the controlling variable), and we take as the minimized quality criterion

$$J = J(f) \to \min_f \tag{1.8}$$

the functional, the value of which depends explicitly or implicitly on the optimized shape f(x,y). The determination of the required projection variable f(x,y) then reduces to taking into account constraints in the form of inequalities, imposed on the total force and moments acting on the punch

$$\Psi_k(P, M_x, M_y) \le 0, \quad k = 1,2,\ldots,m \tag{1.9}$$

An additional restriction on the set of admissible forms of the function f(x,y) arises if we take into account geometrical constraints

$$f \in H \tag{1.10}$$

In relations (1.9) and (1.10) the set H of all admissible shapes of the punch and the functions Ψ_k, which depend on integral functions (1.3), are assumed to be specified. Thus, we can take as relations (1.9), depending on the requirements imposed on the design, for example, the inequalities

$$P \ge P^*, \quad M_x \ge M_x^*, \quad M_y \ge M_y^*$$

written in the form

$$\Psi_1 \equiv P^* - P \le 0, \quad \Psi_2 \equiv M_x^* - M_x \le 0, \quad \Psi_3 \equiv M_y^* - M_y \le 0 \tag{1.11}$$

or the conditions

$$P \ge P^*, \quad M_x \le M_x^*, \quad M_y \le M_y^*$$

represented in the form of constraints (1.9)

$$\Psi_1 \equiv P^* - P \le 0, \quad \Psi_2 \equiv M_x - M_x^* \le 0, \quad \Psi_3 \equiv M_y - M_y^* \le 0 \tag{1.12}$$

where P^*, M_x^*, M_y^* are specified positive constants ($P^* \geq 0$, $M_x^* \geq 0$, $M_y^* \geq 0$). Conditions of continuity and smallness of the gradient can be imposed on the required shape of the punch, i.e.,

$$H = \left\{ f : f(x,y) \in C^0, \quad f(x,y) \geq 0, \quad (\nabla f(x,y))^2 \leq \varepsilon, \quad (x,y) \in \Omega_f \right\} \tag{1.13}$$

where $\varepsilon>0$ is a specified positive constant and C^0 is continuous-function space.

In the contact optimization problem considered here a guaranteed (minimax) approach is used in the uncertainty conditions, based on a search for the “best” solution (the shape of the punch), which minimizes the quality functional of the problem, provided the condition for the “worst” method of external action is satisfied. Hence, to optimize the shape of the rigid punch, indented without friction into an elastic half-space (in the case of incomplete information on the external loads), we will use the system optimization scenario calculated on the basis of the worst case, i.e.,

$$J_{\text{opt}} = \min_{f \in H} \max_{q \in \Lambda_q} J \tag{1.14}$$

2. The total reaction force of the elastic medium and the moments acting on the punch

A direct estimate of the resultant force P and moments M_x and M_y using formulae (1.3) involves considerable difficulties in finding the pressure distribution p(x,y) for arbitrary shape functions $f(x, y)((x, y) \in \Omega_f)$ and external loads $q_x(x, y)$, $q_y(x, y)$, $q_z(x, y)(x, y) \in \Omega_q$. When the solution of the corresponding contact problem of the indentation of a punch with a flat base and the same shape in plan Λ_f is known (or can be obtained analytically or numerically), a more effective way is to determine the forces and moments using the reciprocal theorem.[3–8] We will denote by $p^0(x, y)((x, y) \in \Omega_f)$ and $u^0(x, y)$, $\upsilon^0(x, y)$, $w^0(x, y)((x, y) \in \Omega_q)$ the solution of the contact problem for a plane punch

$$z = \alpha + \beta x + \gamma y, \quad (x,y) \in \Omega_f \tag{2.1}$$

which can be represented in the form[4]

$$\begin{aligned} &p^0(x,y) = \alpha p_\alpha^0 + \beta p_\beta^0 + \gamma p_\gamma^0, \quad u^0(x,y) = \alpha u_\alpha^0 + \beta u_\beta^0 + \gamma u_\gamma^0, \\ &\upsilon^0(x,y) = \alpha \upsilon_\alpha^0 + \beta \upsilon_\beta^0 + \gamma \upsilon_\gamma^0, \quad w^0(x,y) = \alpha w_\alpha^0 + \beta w_\beta^0 + \gamma w_\gamma^0 \end{aligned} \tag{2.2}$$

where α, β and γ are specified constants, and all the quantities with subscripts α, β and γ are functions of x, y and are independent of these constants. Note that in the case of a circular region Ω_f the functions $p^0(x, y)$, $u^0(x, y)$, $\upsilon^0(x, y)$, $w^0(x, y)$ were presented previously in Ref. 4.

If the functions (2.2) for a punch with a plane base are known, the determination of the total forces and moments acting on the punch, having the same shape in plan Ω_q as a plane punch, is considerably simplified. As follows from the reciprocal theorem, pressures p(x,y) will be applied to a punch having the shape z = f(x,y), the resultant of which is found from the formula

$$\begin{aligned} P = &\int_{\Omega_f} f(x,y) p_\alpha^0(x,y) d\Omega_f - \\ &- \int_{\Omega_q} \left[u_\alpha^0(x,y) q_x(x,y) + \upsilon_\alpha^0(x,y) q_y(x,y) + w_\alpha^0(x,y) q_z(x,y) \right] d\Omega_q \end{aligned} \tag{2.3}$$

For the corresponding total moments about the x and y axes, we have the expressions

$$\begin{aligned} M_\xi = &\int_{\Omega_f} f(x,y) p_\delta^0(x,y) d\Omega_f - \\ &- \int_{\Omega_q} \left[u_\delta^0(x,y) q_x(x,y) + \upsilon_\delta^0(x,y) q_y(x,y) + w_\delta^0(x,y) q_z(x,y) \right] d\Omega_q \\ &(\xi,\delta) = (x,\gamma),(y,\beta) \end{aligned} \tag{2.4}$$

3. Optimization of a punch with a circular base

We will assume that the base of the optimized punch has the shape of a circle

$$\Omega_f = \left\{ (x,y) : x^2 + y^2 \leq a^2 \right\} \tag{3.1}$$

and the region where the external load is applied has an arbitrary specified shape (Fig. 1), the point of which $B(x_*, y_*)$ nearest to Ω_f is a distance b from the origin of coordinates, i.e.,

$$b = \sqrt{x_*^2 + y_*^2}, \quad (x_*, y_*) \in \Omega_q, \quad \Omega_f \cap \Omega_q = 0 \tag{3.2}$$

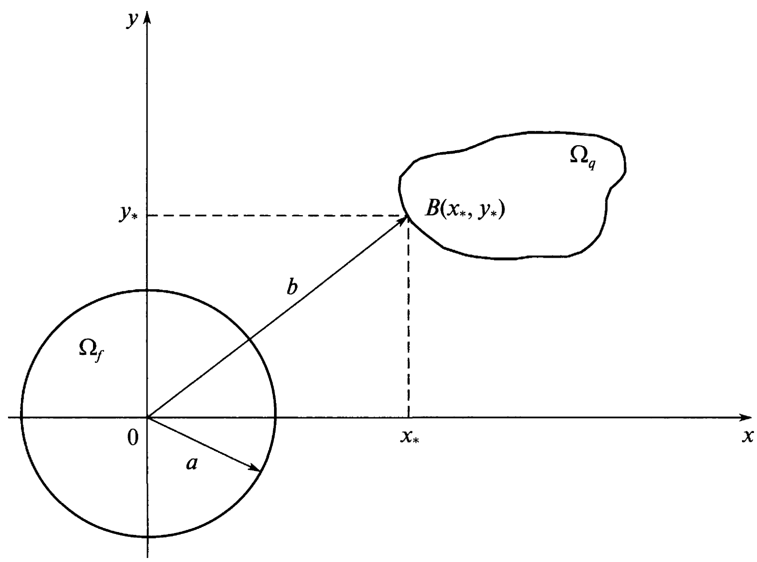

Fig. 1.

where a and b are positive parameters of the problem, where $a < b$. The external forces considered are distributed loads $q_z(x,y)$, which act along the z axis, the resultant of which does not exceed the specified value $Q^* > 0$, i.e.,

$$\Lambda_q = \left\{ q_z : q_z(x,y) \geq 0, \quad \int\limits_{\Omega_q} q_z(x,y) d\Omega_q \leq Q^* \right\} \tag{3.3}$$

The problem of optimizing the mass of the punch reduces to minimizing the functional

$$J(f) = \frac{\rho}{2} \int\limits_{\Omega_f} (\nabla f)^2 d\Omega_f \tag{3.4}$$

with the following constraint imposed on the reaction of the elastic medium

$$P = P_f - P_q \geq P^* \tag{3.5}$$

where

$$P_f = \int\limits_{\Omega_f} f(x,y) p_\alpha^0(x,y) d\Omega_f, \quad P_q = \int\limits_{\Omega_q} w_\alpha^0(x,y) q_z(x,y) d\Omega_q \tag{3.6}$$

$$p_\alpha^0(x,y) = \frac{E}{\pi(1-\nu^2)\sqrt{a^2 - x^2 - y^2}}, \quad (x,y) \in \Omega_f$$

$$w_\alpha^0(x,y) = \frac{2}{\pi} \arcsin \frac{a}{\sqrt{x^2 + y^2}}, \quad (x,y) \in \Omega_q \tag{3.7}$$

Here E and ν are Young's modulus and Poisson's ratio of the elastic medium, ρ is the mass per unit area of the punch, and $P^* > 0$ is a specified positive constant.

Using the guaranteed (minimax) approach employed in this paper, the external load $q_z(x,y)$ is chosen from the admissible set (3.3) of Λ_q in calculating the worst case. Taking into account the inequalities

$$P_f = P_f(f) \geq 0, \quad P_q = P_q(q_z) \geq 0$$

and also the fact that P_q is a linear functional of q_z, while the quantity w_α^0, defined by the second formula of (3.7), is a decreasing function of the distance $r = \sqrt{x^2 + y^2}$, it can be shown that a minimum of the quantity P (3.5) in the set Λ_q (3.3) is attained when

$$q_z = Q^* \delta(x - x_*, y - y_*) \tag{3.8}$$

i.e., when the external load is specified in the form of a concentrated force, applied at the point $(x_*, y_*) \in \Omega_q$ nearest to the region Ω_f. We will denote the Dirac δ-function by δ.

In the case of the worst load (3.8) we will have

$$P = \int\limits_{\Omega_f} f(x,y) p_\alpha^0(x,y) d\Omega_f - \frac{2Q^*}{\pi} \arcsin \frac{a}{b} = P^* \tag{3.9}$$

Introducing the notation $\varphi(r) = p_\alpha^0(x, y)(r = \sqrt{x^2 + y^2})$, we will consider the problem of finding an extremum of the extended Lagrange functional

$$J^L(f) = \frac{\rho}{2}\int_{\Omega_f}(\nabla f)^2 d\Omega_f - \lambda\int_{\Omega_f} f\varphi d\Omega_f \tag{3.10}$$

where λ is the Lagrange multiplier corresponding to constraint (3.9). This problem is axisymmetrical for functional (3.10), while the required shape of the envelope f(r) satisfies the Dirichlet boundary value problem

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = -\frac{\lambda}{\rho}\varphi(r) \tag{3.11}$$

$$f(r) = 0 \text{ when } r = a \tag{3.12}$$

with the additional condition that the required function is limited as r→0. Poisson's equation (3.11) plays the role of the necessary condition for an extremum (Euler's equation) for functional (3.10). Boundary condition (3.12) and the condition that the function f(r) must be finite when r = 0 serve to determine the integration constants. Integrating Eq. (3.11) and choosing the integration constants using the conditions mentioned, we will have

$$f(r) = \frac{\lambda E}{\rho\pi(1-\nu^2)}\left\{\sqrt{a^2 - r^2} - a\ln\left(\frac{a + \sqrt{a^2 - r^2}}{a}\right)\right\}, \quad r \le a \tag{3.13}$$

Hence, the shape of the absolutely rigid punch, indented into the elastic medium, turns out to be axisymmetric, despite the fact that the external loads are not applied symmetrically. Here the depth of penetration is given by the expression

$$f(0) = \frac{\lambda\kappa}{\rho}(1 - \ln 2); \quad \kappa = \frac{E}{\pi(1-\nu^2)}$$

The value of the Lagrange multiplier λ is found using the first equality of (3.7) and relations (3.9) and (3.13). As a result, the optimum form of the punch (convex and axisymmetric) is written in the form

$$f(r) = \frac{1-\nu^2}{Ea(3 - 4\ln 2)}\left(P* + \frac{2Q*}{\pi}\arcsin\frac{a}{b}\right)\left[\sqrt{1 - \left(\frac{r}{a}\right)^2} - \ln\left(1 + \sqrt{1 - \left(\frac{r}{a}\right)^2}\right)\right], \quad r \le a \tag{3.14}$$

We will estimate the pressure under a punch of optimum shape. Using well-known results,[3,4] we will introduce the following dimensionless variables for convenience in obtaining the estimates

$$\tilde{f} = f/a, \quad \tilde{x} = x/a, \quad \tilde{r} = r/a, \quad \tilde{b} = b/a$$

The tilde on the dimensionless quantities will henceforth be omitted. The optimum shape of the punch takes the form

$$f(r) = \lambda_0\left[\sqrt{1 - r^2} - \ln\left(1 + \sqrt{1 - r^2}\right)\right], \quad 0 < r \le 1$$

$$\lambda_0 = \frac{(1-\nu^2)}{E(3 - 4\ln 2)}\left(P* + \frac{2Q*}{\pi}\arcsin\frac{1}{b}\right) > 0 \tag{3.15}$$

When there are no external loads, applied to the region Ω_q, we will have[4] the following expression for the pressure distribution under the punch

$$p_f(r) = \frac{E}{\pi(1-\nu^2)}\left[\frac{C_0}{\sqrt{1 - r^2}} - \int_r^1 \frac{dx}{\sqrt{x^2 - r^2}}\int_0^x\left(\frac{d^2 f}{dt^2} + \frac{1}{t}\frac{df}{dt}\right)\frac{tdt}{\sqrt{x^2 - t^2}}\right] \tag{3.16}$$

$$C_0 = f(0) + \int_0^1 \frac{f(t)dt}{\sqrt{1 - t^2}} \tag{3.17}$$

We convert formula (3.16), using relations (3.15) and (3.17) and the equation for the optimum shape of the punch

$$\frac{d^2 f}{dt^2} + \frac{1}{t}\frac{df}{dt} = -\frac{\lambda_0}{\sqrt{1 - t^2}}, \quad 0 \le t < 1$$

We will have

$$p_f(r) = \frac{\lambda_0 E}{\pi(1-\nu^2)}\left[\frac{\kappa_0}{\sqrt{1 - r^2}} + \int_r^1 \frac{dx}{\sqrt{x^2 - r^2}}\int_0^x \frac{tdt}{\sqrt{x^2 - t^2}\sqrt{1 - t^2}}\right]; \quad 0 \le r < 1, \quad \kappa_0 = 0.563715 \tag{3.18}$$

All the quantities here are strictly positive, and, consequently, the pressure distribution under the punch of optimum shape when there are no additional forces applied to the region Ω_q, is positive.

As was shown in Ref. 3, when external loads are applied to the region Ω_q, additional negative pressures

$$p_q(r) = -\frac{1}{\pi\sqrt{1-r^2}}\int_{\Omega_q} q(\xi,\eta)\frac{\sqrt{\xi^2+\eta^2-1}}{(x-\xi)^2+(y-\eta)^2}d\xi d\eta, \quad r = \sqrt{x^2+y^2} \tag{3.19}$$

will act in the contact area Ω_f. Here the greatest absolute values of the additional pressures are attained when a concentrated force $q = Q^*\delta(\xi-\xi_*, \eta-\eta_*)$, applied at the point closest to the origin of coordinates, is acting, i.e., in the case of the "worst" form of the external load ($\xi_* = b > 1$, $\eta_* = 0$). We have

$$p_q = -\frac{Q^*}{\pi^2}\frac{\sqrt{b^2-1}}{\sqrt{1-r^2}\left((x-b)^2+y^2\right)}, \quad r < 1 \tag{3.20}$$

Estimating the total pressure under the punch, we note that, in view of the fact that the second term on the right-hand side of Eq. (3.18) is positive, the following limit holds

$$p_f(r) \geq \frac{\kappa_0}{\pi(3-4\ln 2)\sqrt{1-r^2}}\left(P^* + \frac{2Q^*}{\pi}\arcsin\frac{1}{b}\right) \tag{3.21}$$

and the negative pressure of greatest absolute value has the following limit

$$|p_q| \leq \frac{Q^*\sqrt{1+b}}{\pi^2\sqrt{1-r^2}(b-1)^{3/2}} \tag{3.22}$$

Consequently, we can represent the limit of the total pressure in the form

$$p(r) = p_f(r) - p_q(r) \geq \frac{1}{\alpha_0\pi^2\sqrt{1-r^2}}[P^* - Q^*\omega(b)] \tag{3.23}$$

$$\omega(b) = \alpha_0\frac{\sqrt{1+b}}{(b-1)^{3/2}} - \beta_0\arcsin\frac{1}{b}, \quad \alpha_0 = \frac{3-4\ln 2}{\pi\kappa_0} = 0.12841, \quad \beta_0 = \frac{2}{\pi} = 0.63662 \tag{3.24}$$

The function $\omega(b)$, as follows from relations (3.24), is positive when $1 < b < b_0 = 1.6235$ and negative when $b_0 < b < \infty$. Hence, when the parameter b varies in the range $b_0 < b < b_\infty$ the pressure p does not change sign and remains positive for any values of $P^* > 0$, $Q^* \geq 0$. If the parameter b satisfies the inequalities $1 < b < b_0$, then in order for the pressure under the punch to be positive it is sufficient to satisfy the condition

$$\frac{P^*}{Q^*} \geq \omega(b) \tag{3.25}$$

Note that these conditions for the pressure to be positive were obtained by using the majorant, and hence they are the sufficient conditions but in general will not be necessary conditions.

4. The optimum shape for combined constraints on the total force and moment

We will now consider the optimization of the shape of a punch having a circular base Ω_f (3.1) with the following combined constraints, imposed, on the force P and the moment M_y

$$P \geq P^*, \quad M_y - M_y^* \geq 0 \tag{4.1}$$

where $P^* > 0$ and $M_y^* > 0$ are specified constans. The area of application of the external load $q_z(x,y)$ has the form

$$\Omega_q = \{(x,y) : b \leq x \leq b+d, \ -c \leq y \leq c\} \tag{4.2}$$

Here b, c and d are specified positive parameters, where $b > a$. As in Section 3, the external loads $q_z(x,y)$ act in the direction of the z axis and have a resultant which does not exceed a specified amount $Q^* > 0$.

Using assumptions (3.1), (3.2) and (4.2) we will consider the problem of optimizing the mass of the shell (3.4) for constraints (4.1), imposed on the reaction of the elastic medium and represented in the form

$$P = P_f - P_q \geq P^*, \quad M_y = M_y^f - M_y^q \geq M_y^* \tag{4.3}$$

where

$$M_y^f = \int_{\Omega_f} f(x,y)p_\beta^0(x,y)d\Omega_f,\quad M_y^q = \int_{\Omega_q} w_\beta^0(x,y)q_z(x,y)d\Omega_q$$

$$p_\beta^0(x,y) = \frac{2Ex}{\pi(1-\nu^2)\sqrt{a^2-x^2-y^2}},\quad (x,y)\in\Omega_f$$

$$w_\beta^0(x,y) = \frac{2x}{\pi}F(r),\quad F(r) = \arcsin\frac{a}{r} - \frac{a}{r^2}\sqrt{r^2-a^2},\quad (x,y)\in\Omega_q \tag{4.4}$$

and, we will use formulae (3.6) and (3.7) to estimate the quantities P_f and P_q.

Taking into account the monotonic decrease in the function

$$w_\beta^0(r,\theta) = \frac{2}{\pi}rF(r)\cos\theta,\quad r > a,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{4.5}$$

as r increases for any θ in the interval $-\pi/2 < \theta < \pi/2$ (θ is the angle measured in the xOy plane anticlockwise from the x axis), and representations (4.3) - (4.5), it can be shown that the "worst" load chosen from the admissible set of loads Λ_q (3.3), has the form of a concentrated load, applied at the point ($x = b$, $y = 0$), i.e.,

$$q_z = Q^*\delta(x-b,y) \tag{4.6}$$

In the case of the worst load (4.6) we will have

$$P = \int_{\Omega_f} f(x,y)p_\alpha^0(x,y)d\Omega_f - \frac{2Q^*}{\pi}\arcsin\frac{a}{b} \ge P^*$$

$$M_y = \int_{\Omega_f} f(x,y)p_\beta^0(x,y)d\Omega_f - \frac{2bQ^*}{\pi}\left(\arcsin\frac{a}{b} - \frac{a}{b}\sqrt{b^2-a^2}\right) \ge M_y^* \tag{4.7}$$

The extended Lagrange functional for the problem of minimizing the mass of the punch (3.4) with constraints (4.7), is given by the expression

$$J^L(f) = \frac{\rho}{2}\int_{\Omega_f}(\nabla f)^2 d\Omega_f - \lambda\int_{\Omega_f} f\varphi d\Omega_f - \mu\int_{\Omega_f} f\psi d\Omega_f \tag{4.8}$$

while the necessary condition for an extremum of functional (4.8) can be written in the form

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} = -\frac{\lambda}{\rho}\varphi(r) - \frac{\mu}{\rho}\psi(r,\theta),\quad 0\le r\le a,\quad 0\le\theta\le 2\pi \tag{4.9}$$

Here λ and μ are Lagrange multipliers corresponding to constraints (4.7), while the functions $\varphi(r)$ and $\psi(r,\theta)$ are given by the expressions

$$\varphi(r) = p_\alpha^0(r) = \frac{E}{\pi(1-\nu^2)\sqrt{a^2-r^2}}$$

$$\psi(r,\theta) = p_\beta^0(r,\theta) = \frac{2Er\cos\theta}{\pi(1-\nu^2)\sqrt{a^2-r^2}} = 2r\varphi(r)\cos\theta \tag{4.10}$$

Poisson's equation (4.9) together with representations (4.10) and the Dirichlet boundary conditions (3.12) constitute the boundary value problem for determining the required shape of the punch. Its solution can be written in the form

$$f(r,\theta) = \lambda\Phi(r) + \mu W(r,\theta) \tag{4.11}$$

where λ and μ are Lagrange multipliers. The function $\lambda\Phi(r)$ is found from the solution of boundary value problem (3.11), (3.12) and is given by expression (3.13). The function $W(r,\theta)$ is the bounded solution of the boundary-value problem

$$\Delta W = -\frac{2r}{\rho}\varphi(r)\cos\theta \tag{4.12}$$

$$(W(r,\theta))_{r=a} = 0,\quad (W(r,\theta))_{r=0} < \infty \tag{4.13}$$

and can be represented in the form

$$W(r,\theta) = \chi(r)\cos\theta$$

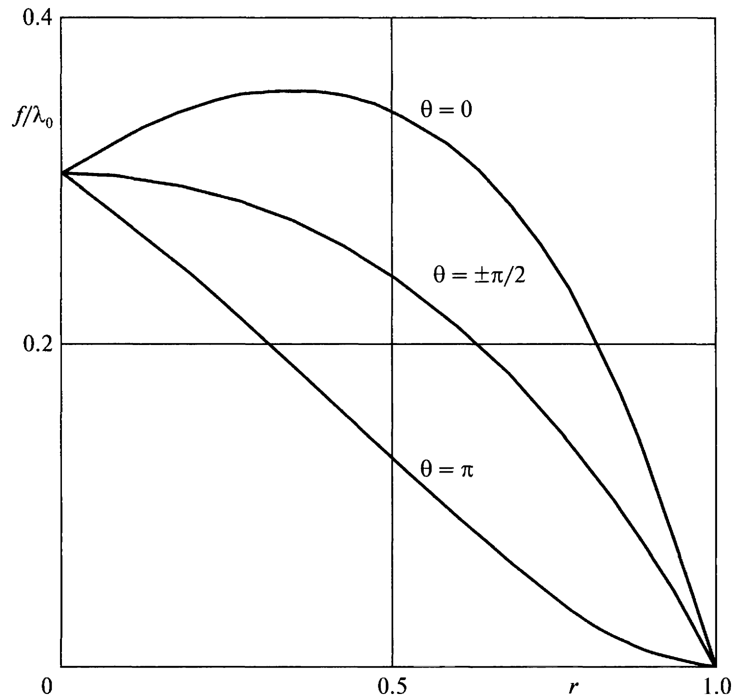

Fig. 2.

The function χ(r) is obtained as the solution of the ordinary differential equation

$$\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}-\frac{1}{r^2}\chi=\frac{2E}{\pi\left(1-\nu^2\right)\rho}\frac{d}{dr}\left(\sqrt{a^2-r^2}\right) \tag{4.14}$$

and is given by the equality

$$r\chi(r)=-\frac{2E}{3\pi\left(1-\nu^2\right)\rho}\left(a^2-r^2\right)^{3/2}+\frac{C}{2}r^2+D \tag{4.15}$$

where C and D are constants, obtained from conditions (4.13). As a result we arrive at the following expression

$$W(r,\theta)=\chi(r)\cos\theta=\frac{2E\cos\theta}{3\pi\left(1-\nu^2\right)\rho r}\left[a\left(a^2-r^2\right)-\left(a^2-r^2\right)^{3/2}\right] \tag{4.16}$$

The Lagrange multipliers λ and μ are found from the system of two linear algebraic equations

$$\begin{aligned}\lambda\int_{\Omega_f}\varphi\Phi d\Omega_f+\mu\int_{\Omega_f}\varphi W d\Omega_f &= P^*+\frac{2Q^*}{\pi}\arcsin\frac{a}{b}\\ \lambda\int_{\Omega_f}\psi\Phi d\Omega_f+\mu\int_{\Omega_f}\psi W d\Omega_f &= M_y^*+\frac{2bQ^*}{\pi}\left(\arcsin\frac{a}{b}-\frac{a}{b}\sqrt{b^2-a^2}\right)\end{aligned} \tag{4.17}$$

In Fig. 2 we show graphs of f/λ_0 against $r(\lambda_0=\lambda E/[\pi\rho(1-\nu^2)])$ for different values of θ for the case when μ = 0.8 λ. Note that the curve for $\theta=\pm\pi/2$ represents the shape of the punch in the case when the constraint on the moments is not imposed $(W(r,\theta)=0)$.

We will now consider another case, when the optimized shape of the punch is found for constraints of the form

$$P\geq P^*,\quad M_y-M^*\leq 0 \tag{4.18}$$

where $P^*>0, M^*>0$ are specified positive constants. In this case the symmetrical shape of the punch, obtained in Section 3, taking into account one constraint imposed on the total force, is the optimum shape. To prove the optimality in the case of constraints (4.18) we note that the minimum of the optimized functional, taking only one of the two constraints into account, will be less or equal to the minimum attained in the case of two constraints, i.e.,

$$\min_f(J)_{P\geq P^*}\leq\min_f(J)_{P\geq P^*,\,M_y\leq M^*} \tag{4.19}$$

Moreover, the optimum symmetrical solution, obtained when only the first constraint of (4.18) is taken into account, satisfies the second condition of (4.18). In fact, the inequality

$$M_y = M_y^f - M_y^q \le M^* \tag{4.20}$$

is satisfied for any positive M*, since f and p_β^0 are symmetrical and antisymmetrical functions of the variable x, respectively, and consequently

$$M_y^f = \int_{\Omega_f} f p_\beta^0 d\Omega_f = 0 \tag{4.21}$$

while the quantity

$$M_y^q = \int_{\Omega_q} w_\beta^0(x,y) q_z(x,y) d\Omega_q \ge 0 \tag{4.22}$$

is non-negative for any forms of $q_z(x,y)$ from relations (3.3) and (4.2). Hence, taking (4.15) - (4.17) into account, the optimum shape of the punch for constraints (4.18) is the symmetrical shape (3.14).

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